ECON 441: Handout 1
301 Review and Externalities
September 9, 2016
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TA Contact Info and Preliminaries

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I will do my best to respond to emails within 24 hours. If you absolutely cannot make my office hours (due to conflicts with another class or work), then I am happy to make an appointment with you. However, I will only make so many appointments in any given week, so email me as early as possible.

Math Review

This is an *applied microeconomics* course, so remembering your micro is absolutely essential to doing well in this class. And since economics is an *applied math* field, this will require remembering some math! Always remember that math is here to help you: reasoning about complex economic phenomena is difficult, and math helps us organize our thinking. If you feel shaky on your calculus, brushing up sooner rather than later will help immensely (this is something I can help you with in office hours). On that note, let's recall two important results in calculus:

Product Rule

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

Example: Let $f(x) = x^2$ and $g(x) = \sin x$. Then:

$$\begin{aligned}(f(x)g(x))' &= f'(x)g(x) + f(x)g'(x) \\ &= 2x \sin x + x^2 \cos x\end{aligned}$$

Chain Rule

$$(f(g(x)))' = f'(g(x))g'(x)$$

Example: Let $f(x) = e^x$ and $g(x) = x^2$. Then (note that $f(g(x)) = e^{x^2}$):

$$\begin{aligned}(f(g(x)))' &= f'(g(x))g'(x) \\ &= e^{x^2}2x\end{aligned}$$

For functions of multiple variables, the Chain Rule for partial derivatives (which we will use often!) is:

$$\frac{\partial}{\partial t}f(x_1, ..., x_n) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial t}$$

Note that if x_j is not a function of t, then $\frac{\partial x_j}{\partial t} = 0$.

Example: Let $u(x(t), y(t)) = x^{\frac{1}{2}}y^{\frac{1}{2}}$ with $x(t) = t^2$ and $y(t) = t$. Then we have

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} \\ &= \frac{1}{2}x^{-\frac{1}{2}}y^{\frac{1}{2}}2t + \frac{1}{2}y^{-\frac{1}{2}}x^{\frac{1}{2}} \\ &= \frac{1}{2}\left(2t^{-1}t^{\frac{1}{2}}t + t^{-\frac{1}{2}}t\right) \\ &= \frac{3}{2}t^{\frac{1}{2}}\end{aligned}$$

which is exactly what we would get if we just substituted $x = t^2$ and $y = t$ into u and differentiated the polynomial $u(t) = t^{\frac{3}{2}}$. It might seem like extra work to use the Chain Rule in this case, but it is often faster (especially on exams!) to use the Chain Rule instead of substituting in long expressions.

Micro Review

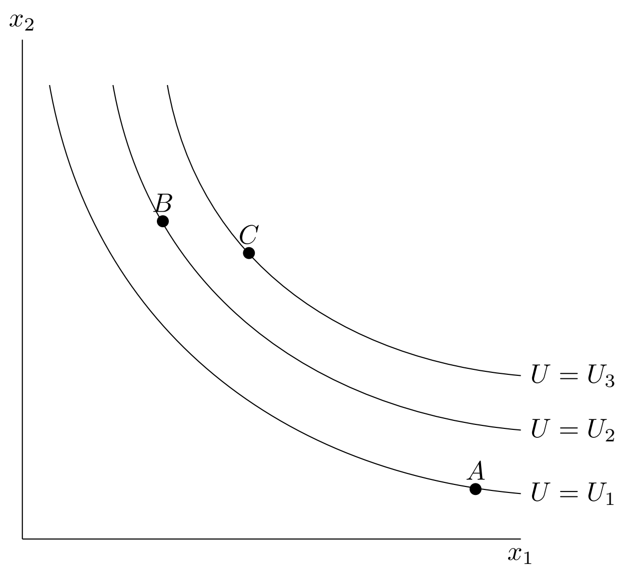
A core idea in microeconomics is that economic agents face *constrained optimization problems*. An agent wants to do something, but they face constraints on what they can do. We rigorously model this with a *utility function* which describes an agent's preferences over bundles of goods and a *budget constraint* which describes what bundles an agent can afford. Let's review how to graphically and analytically solve such a problem.

Indifference Curves

For a given utility function, U, over two goods, x_1, x_2, an *indifference curve* describes the set of all combinations of x_1, x_2 that give the agent some constant level of utility. Mathematically, an indifference curve is a level set (again, just all the points that give a certain level of utility):

$$\{(x_1, x_2) \in \mathbb{R}^2_+ : U(x_1, x_2) = \bar{U}\}$$

Below, three indifference curves are depicted for the specified utility levels of U_1, U_2, U_3. We usually assume that utility levels *increase to the North-East*, thus $U_1 < U_2 < U_3$. Therefore, it is clear that the agent prefers bundle B to bundle A, and prefers bundle C to bundle B.

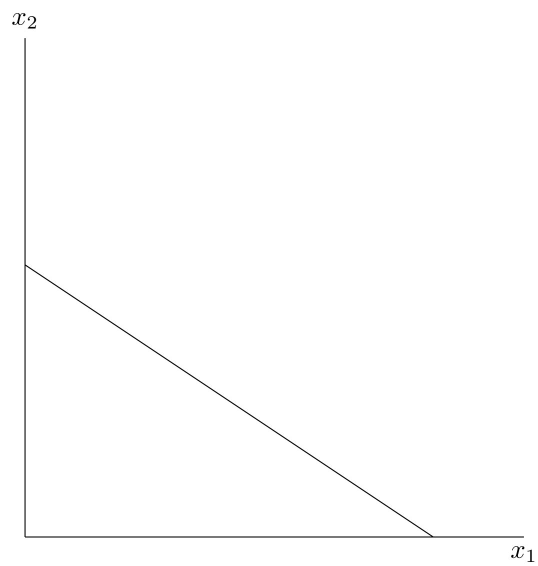


Budget Constraints

Indifference curves describe an agent's preferences, but we are solving a constrained optimization problem, so we need to talk about the constraints. We usually assume that an agent has some income (or endowment), call it w, and he must decide how to allocate this income between the two goods x_1, x_2 at prevailing market prices p_1, p_2. If we assume (as we usually do) that the agent must spend all his income, then this is represented by the equality

$$w = p_1 x_1 + p_2 x_2$$

(or $w = p \cdot x$ if you prefer vector notation). Graphically, this is a line in (x_1, x_2) space:



And it is clear than any point (x_1, x_2) in the triangle defined by this line is affordable by the agent. We say that these points are *feasible*.

What is the slope of this line? Well recall that

$$Slope = \frac{Rise}{Run} = \frac{dx_2}{dx_1}$$

To find $\frac{dx_2}{dx_1}$, we differentiate both sides of the budget constraint $(w = p_1x_1 + p_2x_2)$ with respect to x_1 to get:

$$0 = p_1 + p_2\frac{dx_2}{dx_1} \Longrightarrow \frac{dx_2}{dx_1} = -\frac{p_1}{p_2}$$

Optimization

Having described an agent's preferences and constraint, what particular bundle will he actually choose at the optimum? Specifically, the agent solves:

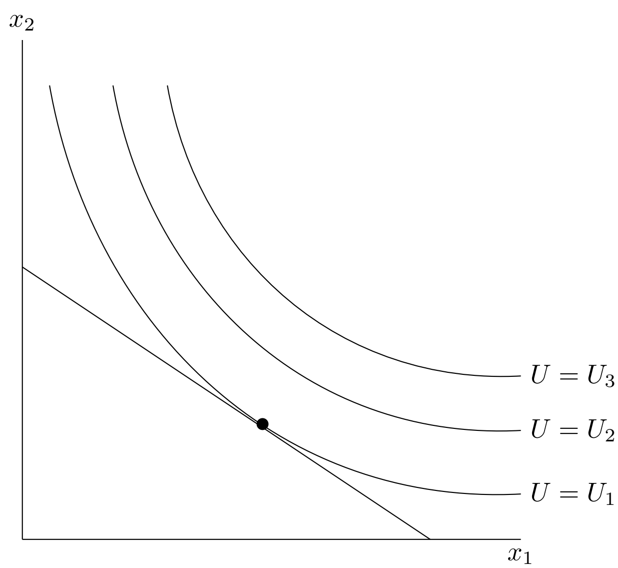
$$\max_{x_1, x_2} U(x_1, x_2)$$

such that

$$w \geq p_1x_1 + p_2x_2$$

and we know that the constraint will hold with equality, because if not, then the agent still has some money that he can spend on either good and increase his utility.

Now in natural language, the problem above is the same as finding the point in the feasible set that lies on the “highest” indifference curve. Graphically, we can see that this is the point where the budget constraint is tangent with an indifference curve:



The agent would certainly like to consume on the U_2 or U_3 lines, but the best he can do while respecting the budget constraint is the tangent point.

MRS=Price Ratio: A fundamental idea in microeconomics is that, *at the optimum, the slope of the indifference curve is the same as the slope of the budget constraint.* We know from above that the slope of the budget constraint is

$$-\frac{p_1}{p_2}$$

So what is the slope of the indifference curve? Again, we need to find:

$$Slope = \frac{Rise}{Run} = \frac{dx_2}{dx_1}$$

Differentiating both sides of the equation that defines the indifference curve ($U_1 = U(x_1, x_2)$), we have (by the Chain Rule!)

$$0 = \frac{\partial U}{\partial x_1} + \frac{\partial U}{\partial x_2}\frac{\partial x_2}{\partial x_1} \Longrightarrow \frac{\partial x_2}{\partial x_1} = -\frac{\partial U}{\partial x_1}/\frac{\partial U}{\partial x_2} \equiv -MRS$$

where MRS is the Marginal Rate of Substitution, or *the amount of good 2 that the agent would accept to give up one unit of good 1.* (Note that the textbook puts a negative in front of the ratio of marginal utilities. As we did in class, it is usually defined without the negative sign. It doesn't matter, just keep track of what you are doing!) It is always a good idea to check units:

$$\frac{\partial U}{\partial x_1}/\frac{\partial U}{\partial x_2} \text{ has units } \frac{Utils}{x_1}/\frac{Utils}{x_2} = x_2 \text{ per } x_1$$

Which were exactly the units we wanted to describe the slope of the indifference curve (the negative sign gets us the fact that the indifference curve has a negative slope in x_1, x_2 space).

Thus, we know that, at the optimum, it must be that the slopes are equal, or:

$$MRS = \frac{p_1}{p_2}$$

This is a great shortcut for solving constrained optimization problems, but let's check it analytically. Again, the problem is:

$$\max_{x_1, x_2} U(x_1, x_2)$$

such that

$$w \geq p_1 x_1 + p_2 x_2$$

We know that the budget constraint will be exhausted, so we can impose that $w = p_1 x_1 + p_2 x_2$ and solve using the method of Lagrange multipliers. Forming the Lagrangian, we have

$$\mathcal{L} = U(x_1, x_2) + \lambda(w - p_1 x_1 - p_2 x_2)$$

with first order necessary conditions

$$\frac{\partial U}{\partial x_1} = \lambda p_1 \Longrightarrow \lambda = \frac{1}{p_1}\frac{\partial U}{\partial x_1}$$
$$\frac{\partial U}{\partial x_2} = \lambda p_2 \Longrightarrow \lambda = \frac{1}{p_2}\frac{\partial U}{\partial x_2}$$

Setting the right-hand sides equal (because they are both equal to λ) and rearranging yields:

$$\frac{\partial U}{\partial x_1} / \frac{\partial U}{\partial x_2} = \frac{p_1}{p_2}$$

which is MRS= $\frac{p_1}{p_2}$!

Math Caveat: The above is predicated on the fact that we knew an *interior solution* existed (this means a positive amount of each good is consumed at the optimum. The agent is not choosing to spend all his income on x_1, for example). This is guaranteed by some assumptions on the utility function. Don't worry about these assumptions, as we will almost always see interior solutions, but do know that they are required.

An Optimization Example

Let $U(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$ with $w = 100$, $p_1 = 1$, and $p_2 = 2$. Then we have that (where MU_i is $\frac{\partial U}{\partial x_i}$) at the optimum,

$$MRS = \frac{MU_1}{MU_2} = \frac{1}{2}\frac{x_2^{\frac{1}{2}}}{x_1^{\frac{1}{2}}} / \frac{1}{2}\frac{x_1^{\frac{1}{2}}}{x_2^{\frac{1}{2}}} = \frac{x_2}{x_1} = \frac{p_1}{p_2} = \frac{1}{2}$$

Rearranging, we have $2x_2 = x_1$. Substituting this into the budget constraint, we have

$$100 = x_1 + 2x_2 = 2x_2 + 2x_2 = 4x_2$$

Thus $x_2 = 25$ and $x_1 = 50$. This makes sense, since x_2 is twice as expensive as x_1, but they enter into the utility function in the same way. Intuitively, $MRS =$ Price Ratio means that the rate at which the agent is willing to trade x_1 for x_2 (MRS) is the same as the rate at which the market is willing to trade x_1 for x_2 (Price Ratio). Another way to think about this is to rearrange:

$$MRS = \frac{p_1}{p_2} \Longrightarrow \frac{MU_1}{p_1} = \frac{MU_2}{p_2}$$

The units of $\frac{MU_i}{p_i}$ are utils per dollar. Thus, the optimization condition is equating "bang per buck" for each good! This must be the case for an interior solution, because if your dollar bought you more utils when consuming good 1 than good 2, you would buy more of good 1 and less of good 2.

We now have two ways of thinking about bundles at which $MRS \neq$ Price Ratio. For example, consider a point where

$$MRS > \frac{p_1}{p_2}$$

First, the MRS is the amount of x_2 you are willing to accept to give up one unit of x_1, and the price ratio is the amount of x_2 that the market will give you for one unit of x_1 (if you don't believe me, check units!). Thus, MRS> $\frac{p_1}{p_2}$ means that you require more x_2 to give up one unit of x_1 than the market will offer. At this point, buying x_2 is a bad deal for you! So you buy less x_2 and more x_1. Second, rearrange to see that

$$MRS > \frac{p_1}{p_2} \implies \frac{MU_1}{p_1} > \frac{MU_2}{p_2}$$

So the "utils per dollar" you get from buying x_1 is greater than from buying x_2, so you should buy more x_1.

Exercise[1]

A graduate student (call him J) lives next to a house of unruly undergrads who like to throw loud parties. The undergrads have a private marginal benefit of $10 - x$, where x is the decibel level of the party. Their cost of increasing the noise is \$5 per decibel (increased electricity costs for their stereo). J is trying to study, so for each decibel he needs to spend \$2 on earplugs.

1. With no government intervention, how loud is the party?
2. What is the socially efficient noise level?
3. What Pigouvian tax would yield the socially efficient level?
4. Suppose J has the right to quiet (which is enforced by the cops). What does the Coase Theorem predict will happen?
5. Suppose the undergrads have the right to party. What does the Coase Theorem predict will happen?

[1] Adapted from an economic theory class taught by Lones Smith, 2016