ECON 441: Handout 12 (and solutions)
Optimal Labor Taxation
December 9, 2016
Joel McMurry

1. Let's consider two types of workers: high-skilled and low-skilled workers. Each workers has a budget constraint:

$$c_i = w_i(1-\tau_i)l_i$$

where c_i is consumption, w_i is wage, τ_i is the tax rate, and l is labor. Let's assume preferences[1] are:

$$U(c_i, l_i) = \frac{1}{1-\sigma}\left(c_i - \frac{l_i^{1+\gamma_i}}{1+\gamma_i}\right)^{1-\sigma}$$

Notice the subscripts i. We have $i = H$ denotes the high-skilled worker and $i = L$ denotes the low-skilled worker. So they differ wage, tax rate, inverse labor supply elasticity γ_i. Assume that $\gamma_L > \gamma_H$, and $w_H > w_L$.

(a) Derive the labor supply curve for each type.

Solution: On your problem set it will be faster to just plug in the budget constraint and solve an unconstrained maximization problem, but here let's set up the Lagrangian:

$$\max_{c,l} \frac{1}{1-\sigma}\left(c_i - \frac{l_i^{1+\gamma_i}}{1+\gamma_i}\right)^{1-\sigma} + \lambda\left[w_i(1-\tau_i)l_i - c_i\right]$$

This has first order conditions:

$$\left(c_i - \frac{l_i^{1+\gamma_i}}{1+\gamma_i}\right)^{-\sigma} - \lambda = 0 \qquad (c_i)$$

$$-{l_i}^{\gamma_i}\left(c_i - \frac{l_i^{1+\gamma_i}}{1+\gamma_i}\right)^{-\sigma} + \lambda w_i(1-\tau_i) = 0 \qquad (l_i)$$

$$w_i(1-\tau_i)l_i - c_i = 0 \qquad (\lambda)$$

[1] These are called Greenwood-Hercowitz-Huffman preferences

Rearranging the first two to get:

$$\left(c_i - \frac{l_i^{1+\gamma_i}}{1+\gamma_i}\right)^{-\sigma} = \lambda$$

$$l_i{}^{\gamma_i}\left(c_i - \frac{l_i^{1+\gamma_i}}{1+\gamma_i}\right)^{-\sigma} \frac{1}{w_i(1-\tau_i)} = \lambda$$

Notice the right-hand sides are the same, thus the left-hand sides are as well. This gives us:

$$\begin{aligned}\left(c_i - \frac{l_i^{1+\gamma_i}}{1+\gamma_i}\right)^{-\sigma} &= l_i{}^{\gamma_i}\left(c_i - \frac{l_i^{1+\gamma_i}}{1+\gamma_i}\right)^{-\sigma} \frac{1}{w_i(1-\tau_i)} \iff \\ &= l_i^{\gamma_i}\frac{1}{w_i(1-\tau_i)} \iff \\ l_i &= (w_i(1-\tau_i))^{1/\gamma_i}\end{aligned}$$

(b) Calculate the labor supply elasticity for each type of worker. Note that labor supply elasticity is:

$$\eta = \frac{\partial l}{\partial(w(1-\tau))}\frac{w(1-\tau)}{l}$$

But we have that

$$\frac{\partial l}{\partial w} = \frac{\partial l}{\partial(w(1-\tau))}(1-\tau)$$

and

$$\frac{\partial l}{\partial(1-\tau)} = \frac{\partial l}{\partial(w(1-\tau))}w$$

Thus,

$$\begin{aligned}\eta &= \frac{\partial l}{\partial(w(1-\tau))}\frac{w(1-\tau)}{l} \\ &= \frac{\partial l}{\partial w}\frac{1}{1-\tau}\frac{w(1-\tau)}{l} \\ &= \frac{\partial l}{\partial w}\frac{w}{l}\end{aligned}$$

and

$$\eta = \frac{\partial l}{\partial(w(1-\tau))}\frac{w(1-\tau)}{l}$$
$$= \frac{\partial l}{\partial(1-\tau)}\frac{1}{w}\frac{w(1-\tau)}{l}$$
$$= \frac{\partial l}{\partial(1-\tau)}\frac{(1-\tau)}{l}$$

This means we can calculate either $\frac{\partial l}{\partial(1-\tau)}\frac{(1-\tau)}{l}$ or $\frac{\partial l}{\partial w}\frac{w}{l}$ and their equivalence will be useful in the next part.

Solution: Let η be labor-supply elasticity. We know that

$$\eta_i = \frac{\partial l_i}{\partial w_i}\frac{w_i}{l_i}$$

Differentiating labor supply with respect to wage, we have

$$\frac{\partial l_i}{\partial w_i} = (1/\gamma_i)\left(w_i(1-\tau_i)\right)^{1/\gamma_i - 1}(1-\tau_i)$$

Plug this into the expression for elasticity along with our result for l_i to get:

$$\eta_i = (1/\gamma_i)\left(w_i(1-\tau_i)\right)^{(1/\gamma_i)-1}(1-\tau_i)\frac{w_i}{l_i}$$
$$= (1/\gamma_i)\left(w_i(1-\tau_i)\right)^{(1/\gamma_i)-1}(1-\tau_i)\frac{w_i}{\left(w_i(1-\tau_i)\right)^{(1/\gamma_i)}}$$
$$= (1/\gamma_i)$$

(c) Recall that an optimal tax system sets $\frac{MU_i}{MR_i}$ equal across all individuals. Compute the marginal revenue of taxing agent i. Hint: $\frac{\partial l_i}{\partial \tau_i} = \frac{\partial l_i}{\partial(1-\tau_i)}$

Solution: We have that

$$\text{Rev}_i = w_i \tau_i l_i$$

Thus, marginal revenue is the derivative of this with respect to the tax:

$$\begin{aligned}
\frac{\partial}{\partial \tau_i} w_i \tau_i l_i &= w_i l_i + w_i \tau_i \frac{\partial l_i}{\partial \tau_i} \\
&= w_i l_i + w_i \tau_i \frac{\partial l_i}{\partial \tau_i} \frac{(1-\tau_i)}{l_i} \frac{l_i}{(1-\tau_i)} \\
&= w_i l_i - w_i \tau_i \frac{\partial l_i}{\partial (1-\tau_i)} \frac{(1-\tau_i)}{l_i} \frac{l_i}{(1-\tau_i)} \\
&= w_i l_i - w_i \tau_i \eta_i \frac{l_i}{(1-\tau_i)} \\
&= w_i l_i \left(1 - \frac{\tau_i (1/\gamma_i)}{(1-\tau_i)}\right)
\end{aligned}$$

Plugging in $l_i = (w_i(1-\tau_i))^{(1/\gamma_i)}$, we have

$$\text{MR}_i = w_i \left(w_i(1-\tau_i)\right)^{(1/\gamma_i)} \left(1 - \frac{\tau_i (1/\gamma_i)}{(1-\tau_i)}\right)$$

(d) We usually find empirically that elasticities of labor supply are highest for high-income workers. Explain why our findings above lead to an equity-efficiency trade-off. First, let's note that MU is with respect to the tax. Investigating this a bit, we have

$$\begin{aligned}
MU &= \frac{\partial}{\partial \tau} U(c, l) \\
&= \frac{\partial}{\partial \tau} U(lw(1-\tau), l) \\
&= U_1 \left(-lw + w(1-\tau)\frac{\partial l}{\partial \tau}\right) + U_2 \frac{\partial l}{\partial \tau} \\
&= -U_1 lw + \frac{\partial l}{\partial \tau} \left(U_1 w(1-\tau) + U_2\right)
\end{aligned}$$

This looks messy, but let's use an envelope condition. Remember what problem the agents are solving:

$$\max_l U(lw(1-\tau), l)$$

This has first order condition:

$$U_1 w(1-\tau) + U_2 = 0$$

So the second part of the MU expression above goes away, and we have

$$MU = -U_1 lw$$

Solution: Using the above, we have

$$\begin{aligned} MU_i &= -U_1 l_i w_i \\ &= -\left(c_i - \frac{l_i^{1+\gamma_i}}{1+\gamma_i}\right)^{-\sigma} l_i w_i \end{aligned}$$

Dividing by $MR_i = w_i l_i \left(1 - \frac{\tau_i(1/\gamma_i)}{(1-\tau_i)}\right)$, we have

$$\frac{MU_i}{MR_i} = -\left(c_i - \frac{l_i^{1+\gamma_i}}{1+\gamma_i}\right)^{-\sigma} \left(1 - \frac{\tau_i(1/\gamma_i)}{(1-\tau_i)}\right)^{-1}$$

Denoting the marginal utility of consumption as U_c, optimal taxation requires

$$\frac{U_{c_L}}{\left(1 - \frac{\tau_L(1/\gamma_L)}{(1-\tau_L)}\right)} = \frac{U_{c_H}}{\left(1 - \frac{\tau_H(1/\gamma_H)}{(1-\tau_H)}\right)}$$

Now equity might suggest that we want to tax to set the marginal utility of consumption equal between both types of workers. But if we did this, then for the equality above to hold, we need

$$\begin{gathered} \left(1 - \frac{\tau_L(1/\gamma_L)}{(1-\tau_L)}\right) = \left(1 - \frac{\tau_H(1/\gamma_H)}{(1-\tau_H)}\right) \Longleftrightarrow \\ \frac{\tau_L(1/\gamma_L)}{(1-\tau_L)} = \frac{\tau_H(1/\gamma_H)}{(1-\tau_H)} \end{gathered}$$

Recall that $1/\gamma_H > 1/\gamma_L$. Let's get clear on how τ affects each side. We check

$$\frac{\partial}{\partial \tau}\frac{\tau}{1-\tau} = \frac{1}{1-\tau} + \frac{\tau}{(1-\tau)^2} > 0$$

Since $(1/\gamma_L) < (1/\gamma_H)$, this implies that the tax rate on low-skill workers should be *higher* than on high-skill workers. This is the efficiency-equity trade-off.